

GENERALIZED MUKAI CONJECTURE FOR SPECIAL FANO VARIETIES

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ABSTRACT. Let X be a Fano variety of dimension n , pseudoinindex i_X and Picard number ρ_X . A generalization of a conjecture of Mukai says that $\rho_X(i_X - 1) \leq n$. We prove that the conjecture holds if: a) X has pseudoinindex $i_X \geq \frac{n+3}{3}$ and either has a fiber type extremal contraction or does not have small extremal contractions b) X has dimension five.

1. INTRODUCTION

Let X be a Fano variety, that is a smooth complex projective variety whose anti-canonical bundle $-K_X$ is ample. We denote with r_X the index of X and with i_X the pseudoinindex of X , defined respectively as

$$r_X = \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some line bundle } L\},$$

$$i_X = \min\{m \in \mathbb{N} \mid -K_X \cdot C = m \text{ for some rational curve } C \subset X\}.$$

In 1988, Mukai [9] proposed the following conjecture:

Conjecture A. Let X be a Fano variety of dimension n . Then

$$\rho_X(r_X - 1) \leq n.$$

A more general conjecture (since $i_X \geq r_X$), which we will consider here, has the following form:

Conjecture B. Let X be a Fano variety of dimension n . Then

$$\rho_X(i_X - 1) \leq n,$$

with equality if and only if $X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}$.

In 1990 Wiśniewski [11] proved that if $i_X > \frac{n+2}{2}$ then $\rho_X = 1$; in that paper he implicitly noticed that the statement of Conjecture B is more natural. In 2002 Bonavero, Casagrande, Debarre and Druel [2] explicitly posed conjecture B and proved it in the following situations: (a) X has dimension 4, (b) X is a toric variety of pseudoinindex $i_X \geq \frac{n+3}{3}$ or of dimension ≤ 7 . In this paper we prove the following

Theorem 1.1. *Let X be a Fano variety of dimension n ; then conjecture B holds in the following cases:*

- (a) $i_X \geq \frac{n+3}{3}$ and X has a fiber type extremal contraction;
- (b) $i_X \geq \frac{n+3}{3}$ and X has not small extremal contractions;
- (c) $n = 5$.

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We use the language of the Minimal Model Program, or Mori theory; therefore for us an extremal contraction is a map with connected fibers from X onto a normal projective variety; such a map contracts all curves in an extremal face of the Kleiman-Mori cone $\overline{NE}(X) \subset N_1(X)$. Remember that, since X is Fano, $NE(X)$ is contained in the half space defined by $\{z \in N_1(X) : K_X z < 0\}$ and so, by the Cone theorem, $NE(X)$ is a polyhedral closed cone.

Note that, while condition (b) is certainly a strong one, condition (a) seems very natural; actually we do not know any example of a Fano variety which does not have fiber type contractions.

We use the typical tools for this kind of problems, in particular the existence of “many” rational curves on X which is a fundamental property of Fano varieties as shown by Mori [8].

We work with families of rational curves, i.e. components of the scheme $\text{Ratcurves}^n(X)$ which parametrizes birational morphisms $\mathbb{P}^1 \rightarrow X$ up to automorphisms of \mathbb{P}^1 , and families of rational 1-cycles, i.e. components of $\text{Chow}(X)$, which we call Chow families; we will denote families of rational curves by capital letters and Chow families by calligraphic letters.

To a family of rational curves V one can associate a Chow family \mathcal{V} , taking the closure of the image of V in $\text{Chow}(X)$ via the natural morphism $\text{Ratcurves}^n(X) \rightarrow \text{Chow}(X)$; if V is an unsplit family, i.e. if V is a proper scheme, then the two notions essentially agree and we can identify V with \mathcal{V} .

Fano varieties are rationally connected, i.e. through every pair of points $x, y \in X$ there exists a rational curve; this was proved in [4] and in [7]. In this paper, as in [1], we use the notion of rational connectedness with respect to some chosen Chow families of rational curves $\mathcal{V}^1, \dots, \mathcal{V}^k$: roughly speaking, X is $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ connected if through every pair of points $x, y \in X$ there passes a connected 1-cycle whose components belong to the families $\mathcal{V}^1, \dots, \mathcal{V}^k$.

To the $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ relation one can associate a proper fibration, called rationally connected fibration, defined on an open set of X , whose fibers are equivalence classes for the relation; this was proved again in [4] and [7].

Using this fact we prove that if X is rationally connected with respect to k unsplit families V^1, \dots, V^k then $\rho_X \leq k$.

Then we show that if X satisfies assumption (a) or (b) of theorem 1.1 then X is rationally connected with respect to $k \leq 3$ unsplit families, and equality holds if and only if $X = (\mathbb{P}^{i_X-1})^3$.

The case of Fano fivefolds is more difficult: we prove that X is rationally connected with respect to a suitable number of proper families, but one of them could be a non unsplit Chow family, so to get the result we have to bound the number of its possible splittings.

2. FAMILIES OF RATIONAL CURVES

We recall some of our basic definitions; our notation is basically consistent with the one in [6] to which we refer the reader.

Let X be a normal projective variety and let $\text{Hom}(\mathbb{P}^1, X)$ be the scheme parametrizing morphisms $f : \mathbb{P}^1 \rightarrow X$; we consider $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X) \subset \text{Hom}(\mathbb{P}^1, X)$, the open subscheme corresponding to those morphisms which are birational onto their image,

and its normalization $\text{Hom}_{bir}^n(\mathbb{P}^1, X)$; the group $\text{Aut}(\mathbb{P}^1)$ acts on $\text{Hom}_{bir}^n(\mathbb{P}^1, X)$ and the quotient exists.

Definition 2.1. The space $\text{Ratcurves}^n(X)$ is the quotient of $\text{Hom}_{bir}^n(\mathbb{P}^1, X)$ by $\text{Aut}(\mathbb{P}^1)$, and the space $\text{Univ}(X)$ is the quotient of the product action of $\text{Aut}(\mathbb{P}^1)$ on $\text{Hom}_{bir}^n(\mathbb{P}^1, X) \times \mathbb{P}^1$.

We have the following commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{bir}^n(\mathbb{P}^1, X) \times \mathbb{P}^1 & \xrightarrow{U} & \text{Univ}(X) \\ \downarrow & & \downarrow p \\ \text{Hom}_{bir}^n(\mathbb{P}^1, X) & \xrightarrow{u} & \text{Ratcurves}^n(X) \end{array}$$

where u and U are principal $\text{Aut}(\mathbb{P}^1)$ -bundles and p is a \mathbb{P}^1 -bundle.

Definition 2.2. We define a family of rational curves to be an irreducible component $V \subset \text{Ratcurves}^n(X)$.

Given a rational curve $f : \mathbb{P}^1 \rightarrow X$ we will call a family of deformations of f any irreducible component $V \subset \text{Ratcurves}^n(X)$ containing $u(f)$.

Given a family of rational curves, we have the following basic diagram:

$$\begin{array}{ccc} p^{-1}(V) =: U & \xrightarrow{i} & X \\ \downarrow p & & \\ V & & \end{array}$$

where i is the map induced by the evaluation $ev : \text{Hom}_{bir}^n(\mathbb{P}^1, X) \times \mathbb{P}^1 \rightarrow X$ and p is a \mathbb{P}^1 -bundle. We define $\text{Locus}(V)$ to be the image of U in X ; we say that V is a covering family if i is dominant, i.e. if $\overline{\text{Locus}(V)} = X$. We will denote by $\deg V$ the anticanonical degree of the family V , i.e. the integer $-K_X \cdot C$ for any curve $C \in V$. If we fix a point $x \in X$, everything can be repeated starting from the scheme $\text{Hom}(\mathbb{P}^1, X; 0 \mapsto x)$ which parametrizes morphisms $f : \mathbb{P}^1 \rightarrow X$ sending $0 \in \mathbb{P}^1$ to x . Again we obtain a commutative diagram

$$(2.1) \quad \begin{array}{ccc} \text{Hom}_{bir}^n(\mathbb{P}^1, X; 0 \mapsto x) \times \mathbb{P}^1 & \xrightarrow{U} & \text{Univ}(X, x) \\ \downarrow & & \downarrow p \\ \text{Hom}_{bir}^n(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{u} & \text{Ratcurves}^n(X, x) \end{array}$$

and, given a family $V \subseteq \text{Ratcurves}^n(X)$, we can consider the subscheme $V \cap \text{Ratcurves}^n(X, x)$ parametrizing curves in V passing through x . We usually denote by V_x a component of this subscheme.

Definition 2.3. Let V be a family of rational curves on X . Then

- (a) V is **unsplit** if it is proper;
- (b) V is **locally unsplit** if for the general $x \in \text{Locus}(V)$ every component V_x is proper.
- (c) V is **generically unsplit** if there is at most a finite number of curves of V passing through two general points of $\text{Locus}(V)$.

Remark 2.4. Note that $(a) \Rightarrow (b) \Rightarrow (c)$.

Proposition 2.5. [6, IV.2.6] *Let X be a smooth projective variety and let V be a family of rational curves.*

Assume either that V is generically unsplit and x is a general point in $\text{Locus}(V)$ or that V is unsplit and x is any point in $\text{Locus}(V)$. Then

- (a) $\dim X + \deg V \leq \dim \text{Locus}(V) + \dim \text{Locus}(V_x) + 1 = \dim V$;
- (b) $\deg V \leq \dim \text{Locus}(V_x) + 1$.

Definition 2.6. We define a **Chow family of rational curves** to be an irreducible component $\mathcal{V} \subset \text{Chow}(X)$ parametrizing rational connected 1-cycles.

Given a Chow family of rational curves, we have a diagram as before, coming from the universal family over $\text{Chow}(X)$.

$$(2.2) \quad \begin{array}{ccc} \mathcal{U} & \xrightarrow{i} & X \\ p \downarrow & & \\ \mathcal{V} & & \end{array}$$

In the diagram i is the map induced by the evaluation and the fibers of p are connected and have rational components. Both i and p are proper (see for instance [6, II.2.2]). By [6, IV.4.10] the family \mathcal{V} defines a proper prerelation in the sense of [6, IV.4.6] (note that schemes and morphisms appearing in that definition are those of the normal form [6, IV.4.4.5]).

Definition 2.7. If V is a family of rational curves we can consider the closure of the image of V in $\text{Chow}(X)$, and call it the **Chow family associated to V** .

Remark 2.8. If V is proper, i.e. if the family is unsplit, then V corresponds to the normalization of the associated Chow family \mathcal{V} ; in particular V itself defines a proper prerelation.

3. CHAINS OF RATIONAL CURVES

Let X be a normal proper variety, $\mathcal{V}^1, \dots, \mathcal{V}^k$ Chow families of rational curves on X and Y a subset of X .

Definition 3.1. We denote by $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ the set of points $x \in X$ such that there exist cycles C_1, \dots, C_k with the following properties:

- C_i belongs to the family \mathcal{V}^i ;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $x \in C_1 \cup \dots \cup C_k$,

i.e. $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ is the set of points which belong to a connected chain of k cycles belonging respectively to the families $\mathcal{V}^1, \dots, \mathcal{V}^k$.

Note that if \mathcal{V} is a Chow family then $\text{Locus}(\mathcal{V})$ is the image of \mathcal{U} in X through i in diagram 2.2, so, since \mathcal{V}, p and i are proper, $\text{Locus}(\mathcal{V})$ is a closed subset of X .

Definition 3.2. We denote by $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles C_1, \dots, C_k with the following properties:

- C_i belongs to the family \mathcal{V}^i ;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $C_1 \cap Y \neq \emptyset$ and $x \in C_k$,

i.e. $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is the set of points that can be joined to Y by a connected chain of k cycles belonging respectively to the families $\mathcal{V}^1, \dots, \mathcal{V}^k$.

Note that $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y \subset \text{Locus}(\mathcal{V}^k)$.

Remark 3.3. If Y is a closed subset, then $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is closed. We prove the statement by induction, since we have

$$\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y = \text{Locus}(\mathcal{V}^k)_{\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^{k-1})_Y}.$$

With the notation of diagram 2.2 let $\mathcal{V}_Y = p(i^{-1}(Y \cap \text{Locus}(\mathcal{V})))$ be the subset of \mathcal{V} parametrizing cycles of \mathcal{V} meeting Y ; $\text{Locus}(\mathcal{V})_Y$ is just $i(p^{-1}(\mathcal{V}_Y))$, so it is closed by the properness of i and p .

Definition 3.4. We denote by $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles C_1, \dots, C_m with the following properties:

- C_i belongs to a family \mathcal{V}^j ;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $C_1 \cap Y \neq \emptyset$ and $x \in C_m$,

i.e. $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is the set of points that can be joined to Y by a connected chain of at most m cycles belonging to the families $\mathcal{V}^1, \dots, \mathcal{V}^k$.

Remark 3.5. Note that

$$\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y = \bigcup_{1 \leq i(j) \leq k} \text{Locus}(\mathcal{V}^{i(1)}, \dots, \mathcal{V}^{i(m)})_Y;$$

in particular, if Y is a closed subset then $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is closed.

Definition 3.6. We define a relation of rational connectedness with respect to $\mathcal{V}^1, \dots, \mathcal{V}^k$ on X in the following way: x and y are in $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ relation if there exists a chain of rational curves in $\mathcal{V}^1, \dots, \mathcal{V}^k$ which joins x and y , i.e. if $y \in \text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_x$ for some m .

Remark 3.7. The $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ relation is nothing but the set theoretic relation $\langle \mathcal{U}_1, \dots, \mathcal{U}_k \rangle$ associated to the proper proalgebraic relation $\text{Chain}(\mathcal{U}_1, \dots, \mathcal{U}_k)$ in the language of [6, IV.4.8].

To the $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ relation we can associate a fibration, at least on an open subset.

Theorem 3.8. [6, IV.4.16] *There exist an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \rightarrow Z^0$ such that*

- (a) *the $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ relation restricts to an equivalence relation on X^0 ;*
- (b) *the fibers of π are equivalence classes for the $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ relation;*
- (c) *for every $z \in Z^0$ any two points in $\pi^{-1}(z)$ can be connected by a chain of at most $2^{\dim X - \dim Z} - 1$ cycles in $\mathcal{V}^1, \dots, \mathcal{V}^k$.*

4. BOUNDING THE PICARD NUMBER OF X

Lemma 4.1. *Let $Y \subset X$ be a closed subset, \mathcal{V} a Chow family of rational curves. Then every curve contained in $\text{Locus}(\mathcal{V})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles parametrized by \mathcal{V} which intersect Y .*

Proof. Let $\mathcal{V}_Y = p(i^{-1}(Y \cap \text{Locus}(\mathcal{V})))$, let $\mathcal{U}_Y = p^{-1}(V_Y)$ and consider the restriction of diagram 2.2

$$\begin{array}{ccc} \mathcal{U}_Y & \xrightarrow{i} & X \\ p \downarrow & & \\ \mathcal{V}_Y & & \end{array}$$

Let C be a curve in $\text{Locus}(\mathcal{V})_Y$ which is not an irreducible component of a cycle parametrized by \mathcal{V} . Then $i^{-1}(C)$ contains an irreducible curve C' which is not contained in a fiber of p and dominates C via i . Let $B = p(C')$ and let S be the surface $p^{-1}(B)$.

Note that there is a curve C'_Y in S which dominates B and such that $i(C'_Y)$ is contained in Y : this is due to the fact that the image via i of every fiber of $p|_S$ meets Y .

By [6, II.4.19] every curve in S is algebraically equivalent to a linear combination with rational coefficients of C'_Y and of the irreducible components of fibers of $p|_S$ (in [6, II.4.19] take $X = S$, $Y = B$ and $Z = C'_Y$).

Thus any curve in $i(S)$, and in particular C , is algebraically, hence numerically, equivalent in $i(\mathcal{U}_Y) = \text{Locus}(\mathcal{V})_Y$ (and hence in X) to a linear combination with rational coefficients of $i_*(C_Y)$ and of irreducible components of cycles parametrized by \mathcal{V}_Y . \square

Corollary 4.2. *Let $Y \subset X$ be a closed subset, $\mathcal{V}^1, \dots, \mathcal{V}^k$ Chow families of rational curves, m a positive integer.*

Then every curve contained in $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles in $\mathcal{V}^1, \dots, \mathcal{V}^k$.

Proof. By 3.5, $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y = \bigcup_{1 \leq i(j) \leq k} \text{Locus}(\mathcal{V}^{i(1)}, \dots, \mathcal{V}^{i(m)})_Y$, so every irreducible component of $\text{ChLocus}_m(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is contained in $\text{Locus}(\mathcal{V}^{i(1)}, \dots, \mathcal{V}^{i(m)})_Y$ for some m -uple $(i(1), \dots, i(m))$.

Then we note that the corollary is true for $\text{Locus}(\mathcal{V}^{i(1)}, \dots, \mathcal{V}^{i(m)})_Y$, applying m times lemma 4.1 with $Y_0 = Y$ and $Y_j = \text{Locus}(\mathcal{V}^{i(1)}, \dots, \mathcal{V}^{i(j)})_Y$. \square

Proposition 4.3. *Let $\mathcal{V}^1, \dots, \mathcal{V}^k$ be Chow families of rational curves on X and let $\pi : X^0 \rightarrow Z^0$ be the $\text{rc}(\mathcal{V}^1, \dots, \mathcal{V}^k)$ fibration.*

Let $Y \subset X$ be a closed subset which dominates Z^0 via π ; then every curve in X is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and irreducible components of cycles in $\mathcal{V}^1, \dots, \mathcal{V}^k$.

Proof. By theorem 3.8 and the assumption, every couple of points in a general fiber of π can be connected by a chain of cycles belonging to $\mathcal{V}^1, \dots, \mathcal{V}^k$ of length at most $M = 2^{\dim X - \dim Z} - 1$. In particular it follows that $\text{ChLocus}_M(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y$ is dense in X and, being closed by remark 3.5, it coincides with X . Then the claim follows from corollary 4.2. \square

Corollary 4.4. *Suppose that X is rationally connected with respect to some Chow families $\mathcal{V}^1, \dots, \mathcal{V}^k$; then every curve in X is numerically equivalent to a linear combination with rational coefficients of irreducible components of cycles in $\mathcal{V}^1, \dots, \mathcal{V}^k$. In particular if X is rationally connected with respect to k unsplit families, then $\rho_X \leq k$.*

Proof. We apply proposition 4.3 with $\pi : X \rightarrow \{*\}$ the contraction of X to a point and Y any point in X . The second part follows from the fact that any cycle parametrized by an unsplit family is irreducible. \square

5. UNSPLIT FAMILIES

The results in the previous section can be enriched if we consider unsplit families of rational curves instead of Chow families.

Lemma 5.1. [10, Lemma 1] *Let $Y \subset X$ be a closed subset, V an unsplit family of rational curves. Then $\text{Locus}(V)_Y$ is closed and every curve contained in $\text{Locus}(V)_Y$ is numerically equivalent to a linear combination with rational coefficients*

$$\lambda C_Y + \mu C_V,$$

where C_Y is a curve in Y , C_V belongs to the family V and $\lambda, \mu \geq 0$.

Note that the improvement with respect to lemma 4.1 is the claim $\lambda \geq 0$.

Corollary 5.2. *Let $R = \mathbb{R}_+[\Gamma]$ be an extremal ray of X , V_Γ a family of deformations of a minimal extremal curve, x a point in $\text{Locus}(V_\Gamma)$ and V an unsplit family of rational curves, independent from V_Γ . Then every curve contained in $\text{Locus}(V_\Gamma, V)_x$ is numerically equivalent to a linear combination with rational coefficients*

$$\lambda C_V + \mu C_\Gamma,$$

where C_V is a curve in V , C_Γ belongs to the family V_Γ and $\lambda, \mu \geq 0$.

Proof. By lemma 5.1, if C is a curve in $\text{Locus}(V_\Gamma, V)_x = \text{Locus}(V)_{\text{Locus}(V_\Gamma)_x}$, then

$$C \equiv \lambda C_\Gamma + \mu C_V,$$

with $\lambda \geq 0$ so we have only to prove that $\mu \geq 0$.

If $\mu < 0$, then we can write $C_\Gamma \equiv \alpha C_V + \beta C$ with $\alpha, \beta \geq 0$, but since C_Γ is extremal, this implies that both $[C]$ and $[C_V]$ belong to R , a contradiction. \square

One of the advantages of using unsplit families is given by the existence of good estimates for the dimension of $\text{Locus}(V^1, \dots, V^k)_x$:

Theorem 5.3. [2, Théorème 5.2] *Let V^1, \dots, V^k be unsplit families of rational curves on X . If the corresponding classes in $N_1(X)$ are independent, then either $\text{Locus}(V^1, \dots, V^k)_x$ is empty or it has dimension greater or equal to $\sum \deg V^i - k$.*

Using the same techniques as in the proof of theorem 5.3 we obtain the following:

Lemma 5.4. *Let $Y \subset X$ be a closed subset and V an unsplit family. Assume that curves contained in Y are numerically independent from curves in V , and that $Y \cap \text{Locus}(V) \neq \emptyset$. Then for a general $y \in Y \cap \text{Locus}(V)$*

- (a) $\dim \text{Locus}(V)_Y \geq \dim(Y \cap \text{Locus}(V)) + \dim \text{Locus}(V_y)$;
- (b) $\dim \text{Locus}(V)_Y \geq \dim Y + \deg V - 1$.

Moreover, if V^1, \dots, V^k are numerically independent unsplit families such that curves contained in Y are numerically independent from curves in V^1, \dots, V^k then either $\text{Locus}(V^1, \dots, V^k)_Y = \emptyset$ or

- (c) $\dim \text{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum \deg V^i - k$.

Proof. We refer to diagram 2. Since V is unsplit, for a point y in $Y \cap \text{Locus}(V)$ we have

$$\dim i^{-1}(y) = \dim V_y = \dim \text{Locus}(V_y) - 1.$$

So, setting $V_Y = p(i^{-1}(Y))$ and $U_Y = p^{-1}(V_Y)$, we have for general $y \in Y \cap \text{Locus}(V)$,

$$\begin{aligned} \dim U_Y &= \dim(Y \cap \text{Locus}(V)) + \dim \text{Locus}(V_y) \geq \\ &\geq \dim Y + \dim \text{Locus}(V) - n + \dim \text{Locus}(V_y) \geq \\ &\geq \dim Y + \deg V - 1. \end{aligned}$$

Since $\text{Locus}(V)_Y = i(U_Y)$, (a) and (b) will follow if we prove that $i : U_Y \rightarrow X$ is generically finite.

To show this we take a point $x \in i(U_Y) \setminus Y$ and we suppose that $i^{-1}(x) \cap U_Y$ contains a curve C' which is not contained in any fiber of p ; let B' be the curve $p(C') \subset V_Y$ and let $\nu : B \rightarrow B'$ be the normalization of B' .

By base change we obtain the following diagram

$$\begin{array}{ccc} S_B & \xrightarrow{j} & X \\ p_B \downarrow & & \\ B & & \end{array}$$

Let C_Y be a curve in S_B which dominates B and whose image via j is contained in Y ; such a curve exists since the image via j of every fiber of p_B meets Y . Now two cases are possible: either $j(C_Y)$ is a point, and therefore we have a one-parameter family of curves passing through two fixed points, contradicting the fact that V is unsplit (see for instance [6, IV.2.3]) or $j(C_Y)$ is a curve in $Y \cap \text{Locus}(V_y)$, so a curve in Y is numerically proportional to a curve parametrized by V , against the assumptions.

To show (c) it is enough to recall that, as already observed in remark 3.3, we have $\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^k)_Y = \text{Locus}(\mathcal{V}^k)_{\text{Locus}(\mathcal{V}^1, \dots, \mathcal{V}^{k-1})_Y}$. \square

Remark 5.5. If in the previous theorem V^1 is not a covering family and $\text{Locus}(V^1, \dots, V^k)_x$ is nonempty, then

$$\dim \text{Locus}(V^1, \dots, V^k)_x \geq \sum \deg V^i - k + 1.$$

In fact $\text{Locus}(V^1, \dots, V^k)_x = \text{Locus}(V^2, \dots, V^k)_{\text{Locus}(V^1_x)}$, and we can apply part (c) of lemma 5.4, recalling that $\dim \text{Locus}(V^1_x) = \deg V^1 - 1$ implies that V^1 is covering (see proposition 2.5).

6. RATIONAL CURVES ON FANO VARIETIES

The geometry of Fano varieties is strongly related to the properties of families of rational curves of low degree. The first result in this direction is a fundamental theorem, due to Mori:

Theorem 6.1. [8] *Through every point of a Fano variety X there exists a rational curve of anticanonical degree $\leq \dim X + 1$.*

Remark 6.2. The families $\{V^i \subset \text{Ratcurves}^n(X)\}$ containing rational curves with degree $\leq n + 1$ are only a finite number, so for at least one index i we have that $\text{Locus}(V^i) = X$. Among these families we choose one with minimal anticanonical

degree, and call it a **minimal dominating family**. Note that every such family is locally unsplit.

A relative version of Mori's theorem is the following

Theorem 6.3. [7, Theorem 2.1] *Let X be a Fano manifold. Suppose that there exist a nonempty open subset X^0 of X , a smooth quasiprojective variety of positive dimension Z^0 and a proper surjective morphism $\pi : X^0 \rightarrow Z^0$. Let z be a general point on Z^0 . Then there exists a rational curve C on X satisfying*

- (a) $C \cap \pi^{-1}(z) \neq \emptyset$;
- (b) C is not contained in $\pi^{-1}(z)$;
- (c) $-K_X \cdot C \leq n + 1$.

Remark 6.4. The families $\{V^i \subset \text{Ratcurves}^n(X)\}$ containing the horizontal curves with degree $\leq n + 1$ are only a finite number, so for at least one index i we have that $\text{Locus}(V^i)$ dominates Z^0 . Among these families we choose one with minimal anticanonical degree, and call it a **minimal horizontal dominating family** for π .

A typical situation where these morphisms arise is the construction of rationally connected fibrations associated to families of rational curves, as we have explained in section 2, or more generally to a finite number of proper connected prerelations, as done in [6, IV.4.16].

Lemma 6.5. *Let X be a Fano variety, and let $\pi : X \dashrightarrow Z$ be the rationally connected fibration associated to m proper connected prerelations on X ; let V be a minimal horizontal dominating family for π . Then*

- (a) *curves parametrized by V are numerically independent from curves contracted by π ;*
- (b) *V is locally unsplit;*
- (c) *if x is a general point in $\text{Locus}(V)$ and F is the fiber containing x , then*

$$\dim(F \cap \text{Locus}(V_x)) = 0.$$

Proof. (a) Since X is normal and Z is proper, the indeterminacy locus E of π in X has codimension ≥ 2 [5, 1.39]. Pull back an ample divisor from Z and observe that it is zero on curves contracted by π . On the other hand it intersects nontrivially curves which are not contracted by π and are not contained in E , like curves of V , since V is dominant.

(b) If for the general $x \in \text{Locus}(V)$ a curve in V_x degenerates into a reducible cycle, then at least one component of this cycle is horizontal, otherwise curves in V would be numerically equivalent to curves in the fibers. But this contradicts the minimality of V among horizontal dominating families.

(c) From lemma 4.1 we know that any curve in $\text{Locus}(V_x)$ is numerically proportional to V , while proposition 4.3 applied to F implies that all curves in F can be written as linear combinations of curves contracted by π . \square

Corollary 6.6. *Let X be a Fano variety, and let $\pi : X \dashrightarrow Z$ be the rationally connected fibration associated to m proper connected prerelations on X ; let V be a minimal horizontal dominating family for π . Then*

$$\deg V \leq \dim Z + 1.$$

Proof. It follows from lemma 6.5 and the fact that $\dim \text{Locus}(V_x) \geq \deg V - 1$. \square

7. SPECIAL FANO VARIETIES OF HIGH PSEUDOINDEX

In this section we will prove Theorem 1.1, (a) and (b).

First of all we will show that Conjecture B is true for a Fano variety X of pseudoindex $i_X \geq \frac{\dim X + 3}{3}$ which has a covering unsplit family of rational curves, then we will prove that this is the case if X is as in (a) or (b).

Proposition 7.1. *Let X be a Fano variety of dimension n and pseudoindex $i_X \geq \frac{n+3}{3}$; if there exists a family V of rational curves which is unsplit and covering then Conjecture B is true for X .*

Proof. Consider the rcV fibration $\pi : X^0 \rightarrow Z^0$: if $\dim Z^0 = 0$ then $\rho_X = 1$ by corollary 4.4 and we conclude. Otherwise take a minimal horizontal dominating family V' ; from lemma 6.5 we know that V'_x is unsplit for general $x \in \text{Locus}(V')$. Then applying lemma 5.4 (b) with $Y = \text{Locus}(V'_x)$ we obtain

$$\begin{aligned} n &\geq \dim \text{Locus}(V)_{\text{Locus}(V'_x)} \geq \\ &\geq \dim \text{Locus}(V'_x) + \deg V - 1 \geq \\ &\geq \deg V' + \deg V - 2 \end{aligned}$$

so $\deg V' \leq 2i_X - 1$ and therefore V' is unsplit.

Take the rc(V, V') fibration $\pi' : X' \rightarrow Z'$: if $\dim Z' = 0$ then from corollary 4.4 we have $\rho_X = 2$ and we conclude, otherwise take a minimal dominating family V'' with respect to π' .

For general $x \in \text{Locus}(V'')$, denote by F the fiber of π' containing x : then F is an equivalence class with respect to the rc(V, V') relation, so $F \supseteq \text{Locus}(V, V')_y$ for some y ; then theorem 5.3 implies

$$\dim F \geq \deg V + \deg V' - 2 \geq 2i_X - 2.$$

By lemma 6.5 we have $\dim(\text{Locus}(V''_x) \cap F) = 0$, so

$$n \geq \dim F + \dim \text{Locus}(V''_x) \geq 2i_X - 2 + \deg V'' - 1,$$

that is

$$\deg V'' \leq n + 3 - 2i_X \leq i_X.$$

This is impossible unless $\deg V = \deg V' = \deg V'' = i_X$ and $\dim \text{Locus}(V_x) = \dim \text{Locus}(V'_x) = \dim \text{Locus}(V''_x) = i_X - 1$. Proposition 2.5 implies that all these families are covering, so we can apply [10, Theorem 1] to obtain that $X \simeq (\mathbb{P}^{i_X-1})^3$. \square

Theorem 7.2. *Let X be a Fano variety of dimension n and pseudoindex $i_X \geq \frac{n+3}{3}$. If X has a fiber type extremal contraction or has not small contractions then there exists a covering unsplit family V of rational curves.*

Proof. First of all suppose that there exists a fiber type contraction $\varphi : X \rightarrow W$; let V_φ be a minimal horizontal dominating family for φ ; from corollary 6.6 we know that $\deg V_\varphi \leq \dim W + 1$. Let F be a general fiber of φ ; we have that

$$\dim F \leq \dim X - \deg V_\varphi + 1 \leq 2i_X - 2.$$

By adjunction we have $K_F = (K_X)_F$, so F is a Fano variety; in particular there exists a minimal dominating family V_F of degree $\leq \dim F + 1 \leq 2i_X - 1$.

This means that through a general point of X there passes a curve of degree $\leq 2i_X - 1$, and since the families of rational curves with bounded degree are a

finite number, one of them must be covering; the bound on the degree implies that this family is also unsplit.

Suppose now that all the extremal contractions of X are divisorial and, by contradiction, that there does not exist any unsplit covering family of rational curves. Let V be a minimal dominating family of rational curves; since we are assuming that V is not unsplit we have $\deg V \geq 2i_X$.

Consider the Chow family \mathcal{V} associated to V : since $\deg V \leq n+1 < 3i_X$, reducible cycles in \mathcal{V} split into exactly two irreducible components. To each one of them we associate the corresponding irreducible component of $\text{Ratcurves}^n(X)$, which is an unsplit family.

We denote by \mathcal{B} the finite set of pairs of families (W^i, W^{i+1}) satisfying:

- $[W^i]$ is numerically independent from $[W^{i+1}]$;
- $[W^i] + [W^{i+1}] = [V]$;
- W^i and W^{i+1} contain irreducible components of cycles of \mathcal{V} .

Consider now the $\text{rc}\mathcal{V}$ fibration $\pi : X^0 \rightarrow Z^0$.

Claim. $\dim Z^0 = 0$.

Suppose by contradiction that Z^0 has positive dimension, and take V' a minimal horizontal dominating family for π ; we know from lemma 6.5 (c) that for a general fiber F we have

$$\dim \text{Locus}(V'_x) + \dim F \leq n,$$

which implies

$$\deg V' \leq n+1 - \dim F \leq n - 2i_X + 2 < i_X,$$

a contradiction which proves the claim.

As a corollary we obtain that $N_1(X)$ is generated as a vector space by the numerical classes of the irreducible components of cycles in \mathcal{V} (proposition 4.3).

Note that if $[V]$ is extremal in $NE(X)$, then all the irreducible components of cycles in \mathcal{V} are numerically proportional to $[V]$ and $\rho_X = 1$, so we can assume that $[V]$ is not extremal.

Take now $R_1 = \mathbb{R}_+[C_1]$ to be a divisorial extremal ray of X , let E_1 be its exceptional locus and V^1 an unsplit family of deformations of a minimal extremal rational curve C_1 .

First of all we claim that $E_1 \cdot V = 0$; otherwise for a general $x \in X$ the set $\text{Locus}(V^1)_{\text{Locus}(V_x)}$ would be nonempty, so by lemma 5.4 and proposition 2.5

$$\dim \text{Locus}(V^1)_{\text{Locus}(V_x)} \geq \dim \text{Locus}(V_x) + \deg V^1 - 1 \geq 3i_X - 2 > \dim X.$$

In particular we find a pair $(W^1, W^2) \in \mathcal{B}$ such that $E_1 \cdot W^1 < 0$ and $E_1 \cdot W^2 > 0$.

By corollary 5.2, if $[W^1] \neq [V^1]$ then the class of every curve in $\text{Locus}(V^1, W^1)_x$ can be written as a linear combination with positive coefficients of $[V^1]$ and $[W^1]$, so, for $x \in \text{Locus}(W^1) \cap \text{Locus}(W^2)$,

$$\dim(\text{Locus}(W^2_x) \cap \text{Locus}(V^1, W^1)_x) = 0;$$

on the other hand, if $[W^1] \neq [V^1]$, by remark 5.5 we have

$$\dim \text{Locus}(V^1, W^1)_x \geq 2i_X - 1,$$

and therefore

$$\dim(\text{Locus}(W_x^2) \cap \text{Locus}(V^1, W^1)_x) \geq 3i_X - 2 - n > 0.$$

We thus get a contradiction, unless $[W^1] = [V^1]$.

Note that this argument also shows that for all $i \neq 1, 2$ we have $E_1 \cdot W^i = 0$.

Since X is Fano and E_1 is effective there exists an extremal ray R_2 on which E_1 is positive (this is due to the fact that every effective curve on a Fano manifold can be written as a linear combination with positive coefficients of extremal curves: see [3, Lemma 2]); let E_2 be the exceptional locus of R_2 .

We repeat the same argument and we find a pair (W^3, W^4) such that $[V^2] = [W^3]$ and $E_2 \cdot W^4 > 0$.

If the plane Π_1 spanned in $N_1(X)$ by the classes $[V]$ and $[V^1]$ is different from the plane Π_2 spanned by $[V]$ and $[V^2]$, then $[V^1]$, $[V^2]$ and $[W^4]$ are independent, and $\text{Locus}(W^4, V^2, V^1)_x$ is nonempty for every $x \in \text{Locus}(W^4)$. By remark 5.5 we get

$$\dim \text{Locus}(W^4, V^2, V^1)_x \geq 3i_X - 2 > n,$$

a contradiction.

So we suppose that $\Pi_1 = \Pi_2 := \Pi$ and we choose a basis of $N_1(X)$ formed by $[V^1]$, $[V]$ and by classes $[W^i]$ not contained in Π .

Since the divisors E_1 and E_2 are zero on all the elements of the basis but $[V^1]$, they are proportional in $N^1(X)$; but $E_1 \cdot V^1 < 0$ and $E_2 \cdot V^1 > 0$, so $E_1 = -kE_2$ with $k > 0$. One can now compute the intersection number of E_1 and E_2 with any curve which meets $E_1 \cup E_2$ without being contained in it, and this leads to a contradiction. \square

8. FANO FIVEFOLDS WITH A COVERING UNSPLIT FAMILY

This section and the following one are devoted to the proof of Theorem 1.1 (c).

Let X be a Fano variety of dimension 5 and let $V \subseteq \text{Ratcurves}^n(X)$ be a minimal dominating family; by remark 6.2 we have that $\deg V \leq 6$ and V_x is unsplit for a general $x \in X$.

If $\deg V = 6$ then $X = \text{Locus}(V_x)$ and $\rho_X = 1$ by lemma 4.1, therefore we can assume $\deg V \leq 5$.

First of all we note that if $i_X \geq 3$, then V is unsplit; moreover in this case we can apply proposition 7.1 and obtain the result, so from now on we assume that $i_X = 2$ (and we thus have to prove that $\rho_X \leq 5$).

We divide the proof into two main cases: in this section we will deal with the case in which V is **unsplit**, while in the next one we will assume that V is **not unsplit**.

Consider the rcV fibration $\pi : X^0 \rightarrow Z^0$: if $\dim Z^0 = 0$ then $\rho_X = 1$ by corollary 4.4 and we conclude; otherwise take a minimal horizontal dominating family V' .

Case 1. Any minimal horizontal dominating family V' is not unsplit.

Note that in this situation $\deg V' \geq 4$, so $\dim \text{Locus}(V'_x) \geq 3$; in particular, since V' is horizontal and dominates Z^0 , we have also $\dim Z^0 \geq 3$.

If $\dim Z^0 = 3$ take a general point $x \in \text{Locus}(V')$, so that V'_x is unsplit. Note that $Y = \text{Locus}(V'_x)$ dominates Z^0 , so we can apply proposition 4.3 to get $\rho_X = 2$.

If $\dim Z^0 = 4$ consider the rc(V, V') fibration $\pi' : X' \rightarrow Z'$.

Claim. $\dim Z' = 0$.

Assume that this is not the case and denote by F' a general fiber of π' . Then there exists a minimal horizontal dominating family V'' satisfying

$$\begin{aligned} 0 = \dim(F' \cap \text{Locus}(V''_x)) &\geq \dim F' + \dim \text{Locus}(V''_x) - 5 \\ &\geq 4 + \dim \text{Locus}(V''_x) - 5 \\ &\geq \deg V'' - 2 \end{aligned}$$

for every $x \in F' \cap \text{Locus}(V'')$. Thus $\deg V'' = 2$ and $\dim \text{Locus}(V''_x) = 1$, so by proposition 2.5 V'' is covering. Since V'' is horizontal also with respect to the fibration π this contradicts the minimality of V' , thus the claim is proved.

From corollary 6.6 it follows that $\deg V' \leq 5$, so every reducible cycle in \mathcal{V}' splits into exactly two irreducible components; moreover the family of deformations of each component is unsplit and non covering because of the minimality of V' .

Consider the pairs (W^i, W^{i+1}) of unsplit families satisfying

- $[W^i] + [W^{i+1}] = [V']$,
- W^i and W^{i+1} contain irreducible components of a cycle in \mathcal{V}' ,

and let \mathcal{B} be the set of these pairs.

If the numerical class of every pair in \mathcal{B} lies in the plane $\Pi \subseteq N_1(X)$ spanned by $[V]$ and $[V']$ then, by corollary 4.4 we have that $\rho_X = 2$ and we are done.

Assume therefore by contradiction that there exists a pair $(W^1, W^2) \in \mathcal{B}$ whose classes don't lie in Π , call Π' the plane spanned by $[W^1]$ and $[W^2]$ and set

$$\mathcal{B}_{\Pi, \Pi'} = \{(W^i, W^{i+1}) \in \mathcal{B} \mid [W^i], [W^{i+1}] \in \Pi, \Pi' \text{ and } [W^i], [W^{i+1}] \neq [\lambda V]\}.$$

For every $(W^i, W^{i+1}) \in \mathcal{B}_{\Pi, \Pi'}$, for every cycle $C_i + C_{i+1} \in W^i + W^{i+1}$ and for every point $x \in C_i$ we consider $\text{Locus}(W^i, V, W^{i+1})_x$: by remark 5.5, we have $\dim \text{Locus}(W^i, V, W^{i+1})_x \geq 4$; since W^{i+1} is not covering every irreducible component of $\text{Locus}(W^i, V, W^{i+1})_x$ is an effective divisor on X , which is contained in $\text{Locus}(W^{i+1})$. Since W^{i+1} does not dominate Z^0 , the intersection of any of these divisors with V is zero.

We claim that the intersection of any of these divisors with V' is also zero.

In fact, if $D = \text{Locus}(W^i, V, W^{i+1})_x$ is such that $D \cdot V' > 0$, then every curve in V' intersects $\text{Locus}(W^{i+1})$. Since V is covering we have

$$\text{Locus}(V)_{\text{Locus}(V'_x)} \supseteq \text{Locus}(V'_x),$$

so

$$\text{Locus}(V, W^{i+1})_{\text{Locus}(V'_x)} \neq \emptyset;$$

we apply lemma 5.4 (c) and we obtain that $\dim \text{Locus}(V, W^{i+1})_{\text{Locus}(V'_x)} = 5$, which implies that W^{i+1} is covering, a contradiction.

Obviously we can repeat the same argument with $\text{Locus}(W^{i+1}, V, W^i)_x$ for every $x \in C_{i+1}$, and we obtain effective divisors which are contained in $\text{Locus}(W^i)$ and whose intersection with V and V' is zero.

Call T the union of all these divisors. Now take a point $y \in X \setminus T$; since X is $\text{rc}(V, \mathcal{V}')$ connected, y can be joined to T by a chain of curves in V and cycles in \mathcal{V}' . In particular there exists a cycle Γ either in V or in \mathcal{V}' which intersects T but is not contained in it, and since every component of T has intersection zero with V and V' , it must be of the form $C_3 + C_4$, with $(W^3, W^4) \in \mathcal{B}$ and $[W^3], [W^4] \notin \Pi, \Pi'$.

So, up to exchange W^3 and W^4 , there exists a component D of T such that $D \cdot W^3 > 0$; then $\text{Locus}(W^3)_D$ is nonempty and, by lemma 5.4 (b),

$$\dim \text{Locus}(W^3)_D \geq \dim D + \deg W^3 - 1 \geq 5$$

and W^3 is covering, a contradiction.

Case 2. One minimal horizontal dominating family V' is unsplit.

Consider the $\text{rc}(V, V')$ fibration $\pi' : X' \rightarrow Z'$; if Z' is a point then $\rho_X = 2$ and we conclude, otherwise take a minimal horizontal dominating family V'' .

If V'' is not unsplit then $\deg V'' \geq 4$, so $\dim \text{Locus}(V''_x) \geq 3$; moreover, since $\dim Z' \leq 3$, $\text{Locus}(V''_x)$ dominates Z' .

Take a general point $x \in \text{Locus}(V'')$, so that V''_x is unsplit and apply proposition 4.3 with V, V' and $Y = \text{Locus}(V''_x)$ to obtain $\rho_X = 3$.

If V'' is unsplit we can take the $\text{rc}(V, V', V'')$ fibration $\pi'' : X'' \rightarrow Z''$: either Z'' is a point or every minimal horizontal dominating family is unsplit. We consider the new fibration and we repeat the same argument. Finally we find at most five independent unsplit families on X such that X is rationally connected with respect to them, so $\rho_X \leq 5$ by corollary 4.4.

If there are exactly five independent families, then they must be covering and of degree 2 and from [10] we conclude that $X \simeq (\mathbb{P}^1)^5$.

9. FANO FIVEFOLDS WITHOUT A COVERING UNSPLIT FAMILY

We assume now that every minimal dominating family V of X is not unsplit, which implies that $\deg V \geq 4$.

By the discussion at the beginning of the previous section we can also assume that $i_X = 2$ and that $\deg V \leq 5$, so every reducible cycle in the associated Chow family \mathcal{V} splits into exactly two irreducible components; moreover any family of deformations of each component is unsplit and non covering because of the minimality of V .

Consider the pairs (W^i, W^{i+1}) of unsplit families satisfying

- $[W^i] + [W^{i+1}] = [V]$,
- W^i and W^{i+1} contain irreducible components of a cycle in \mathcal{V} ,

and let \mathcal{B} be the set of these pairs.

Claim. If $\deg V = 5$ then $\rho_X = 1$.

Assume by contradiction that $\deg V = 5$ and $\rho_X \geq 2$.

Suppose that all the irreducible components of cycles in \mathcal{V} are numerically proportional to V , and consider the $\text{rc}\mathcal{V}$ fibration $\pi : X^0 \rightarrow Z^0$.

Now, either Z^0 is a point and in our assumptions $\rho_X = 1$ by corollary 4.4, or there exists a minimal horizontal dominating family V' ; then for a general $x \in \text{Locus}(V')$, if F is the fiber through x , we know from lemma 6.5 that

$$\dim \text{Locus}(V'_x) + \dim F \leq 5,$$

and since $\dim F \geq \deg V - 1 = 4$ we have $\deg V' - 1 \leq \dim \text{Locus}(V'_x) \leq 1$, forcing $\deg V' = 2$ and $\dim \text{Locus}(V'_x) = 1$; hence by proposition 2.5 V' is covering, against the assumptions.

So there exists a pair $(W^1, W^2) \in \mathcal{B}$ such that $[W^1] \neq [\alpha V]$.

Let D be an irreducible component of $\text{Locus}(V_x)$ for a general $x \in X$; since V is locally unsplit we have $N_1(D) = \langle [V] \rangle$.

By proposition 2.5, $\dim D \geq \deg V - 1 \geq 4$; as we are assuming $\rho_X \geq 2$ it cannot be $D = X$, so D is an effective divisor.

If $D.V = 0$ then D would be negative on at least a family W^i and so it would contain curves in W^i , contradicting the fact that $N_1(D) = \langle [V] \rangle$.

If else $D.V > 0$, then either $D.W^1 > 0$ or $D.W^2 > 0$; but in this case either $\text{Locus}(W_x^1) \cap D$ or $\text{Locus}(W_x^2) \cap D$ would be nonempty. Since W^i is not covering we have $\dim \text{Locus}(W_x^i) \geq 2$, therefore $\dim(\text{Locus}(W_x^1) \cap D) \geq 1$, against the fact that $N_1(\text{Locus}(W_x^i)) = \langle [W^i] \rangle$. So the claim is proved and we can assume from now on that $\deg V = 4$.

Consider the $\text{rc}\mathcal{V}$ fibration $\pi : X^0 \rightarrow Z^0$.

Case 1 $\dim Z_0 > 0$.

In this case we actually prove that $\rho_X = 2$.

Choose V' to be a minimal horizontal dominating family for π ; again we know that

$$\dim \text{Locus}(V'_x) + \dim F \leq 5,$$

but in this case $\dim F \geq 3$ so $\deg V' - 1 \leq \dim \text{Locus}(V'_x) \leq 2$.

On the other hand $\dim \text{Locus}(V'_x) \geq \deg V' \geq 2$, since otherwise V' would be covering and of degree 2 by 2.5, contradicting the minimality of V .

It follows that $\dim F = 3$, $\dim \text{Locus}(V'_x) = 2$ and $\deg V' = 2$, so V' is unsplit and $\dim \text{Locus}(V') = 4$.

Moreover, since $\text{Locus}(V'_x)$ meets the general fiber of π , then X is $\text{rc}(\mathcal{V}, V')$ connected.

Let Π be the plane spanned by $[V]$ and $[V']$ and let

$$\mathcal{B}_\Pi = \{(W^i, W^{i+1}) \in \mathcal{B} \mid [W^i] \text{ and } [W^{i+1}] \in \Pi\}.$$

If $\mathcal{B}_\Pi = \mathcal{B}$ then we have $\rho_X = 2$ by corollary 4.4.

Suppose that this is not the case and let D' be an irreducible component of $\text{Locus}(V')$.

Since D' does not contain the general fiber F of π and the general F coincides with $\text{Locus}(V_x)$ for some x , there exists a curve of V meeting D' but not entirely contained in it; therefore $D'.V > 0$.

Let $V'_{D'}$ be the closed subfamily of V' such that $\text{Locus}(V'_{D'}) = D'$; by lemma 5.4

(a) $\dim \text{Locus}(V'_{D'})_{D' \cap F} \geq 4$ i.e. $\text{Locus}(V'_{D'})_{D' \cap F} = D'$.

Since $N_1(D' \cap F) = \langle [V] \rangle$, lemma 4.1 implies that $N_1(D') = \langle [V], [V'] \rangle$.

Let (W^1, W^2) be a pair in $\mathcal{B} \setminus \mathcal{B}_\Pi$; since $D'.V > 0$ either $D'.W^1 > 0$ or $D'.W^2 > 0$, so we can assume $\text{Locus}(W^1)_{D'} \neq \emptyset$; but this implies by lemma 5.4 that $\dim \text{Locus}(W^1)_{D'} \geq 5$, and therefore that W^1 is covering, a contradiction.

Case 2 $\dim Z_0 = 0$ i.e. X is $\text{rc}\mathcal{V}$ connected.

In this case by corollary 4.4 $N_1(X)$ is generated as a vector space by the numerical classes of the irreducible components of cycles in \mathcal{V} .

We want to show that $\rho_X \leq 3$, so by contradiction we assume that there exist three pairs (W^1, W^2) , (W^3, W^4) and (W^5, W^6) in \mathcal{B} whose classes generate a four

dimensional vector space inside $N_1(X)$.

Let $\Pi \subset N_1(X)$ be the plane generated by $[W^1]$ and $[W^2]$, and let

$$\mathcal{B}_\Pi = \{(W^i, W^{i+1}) \in \mathcal{B} \mid [W^i] \text{ and } [W^{i+1}] \in \Pi\}.$$

For every pair $(W^i, W^{i+1}) \in \mathcal{B}_\Pi$ let $\{D_k^i\}$ be the components of $\text{Locus}(W^i)$ which intersect $\text{Locus}(W^{i+1})$ and let $\{D_j^{i+1}\}$ be the components of $\text{Locus}(W^{i+1})$ which intersect $\text{Locus}(W^i)$.

Let us note that, by proposition 2.5, every component of $\text{Locus}(W^i)$ has dimension greater than three, so, since the families W^i are not covering, we have $\dim D_k^i = 3$ or 4.

Case 2a For every i and every k there exists j such that $D_k^i = D_j^{i+1}$ and viceversa.

If $\dim D_k^i = 3$ then by proposition 2.5 $\dim \text{Locus}(W_x^i) = 3$ and D_k^i is a component of $\text{Locus}(W_x^i)$ for some x , so $N_1(D_k^i) = \langle [W^i] \rangle$; but since $D_k^i = D_j^{i+1}$ for some j , we have also $N_1(D_k^i) = \langle [W^{i+1}] \rangle$, a contradiction.

So we can assume that D_k^i is a divisor for every k ; moreover D_k^i is a component of $\text{Locus}(W^i)_{\text{Locus}(W^{i+1})}$, and so $N_1(D_k^i) = \langle [W^i], [W^{i+1}] \rangle$.

Let us consider the intersection number of one of these divisors, say $D_1^1 =: D$, with the family V ; if $D \cdot V > 0$ then, up to exchange W^3 and W^4 , we have $D \cdot W^3 > 0$. By lemma 5.4, since $\text{Locus}(W^3)_D$ is nonempty, we have $\dim \text{Locus}(W^3)_D = 5$, a contradiction since W^3 is not covering.

Therefore $D \cdot V = 0$, hence $D \cdot W^i < 0$ for some i ; since $N_1(D)$ is generated by the classes of W^1 and W^2 , the class of W^i must belong to the plane Π .

In particular for every pair $(W^i, W^{i+1}) \in \mathcal{B}_\Pi$ we have that $(D \cdot W^i)(D \cdot W^{i+1}) < 0$, yielding that

$$D = \text{Locus}(W^i) = \text{Locus}(W^{i+1}).$$

Let now x be a point outside D and let z be a point of D ; since X is $\text{rc}\mathcal{V}$ connected there exists a chain of cycles in \mathcal{V} which connects x and z ; let Γ be the first irreducible component of one of these chains which meets D .

Since $D \cdot V = 0$ then Γ cannot belong to V or to a family which is proportional to V . Moreover, since $\Gamma \not\subset D$ then Γ does not belong to a family whose class is contained in the plane Π .

Therefore Γ belongs to a family W^i whose class is not in Π ; we can thus apply lemma 5.4 and obtain $\dim \text{Locus}(W^i)_D = 5$, a contradiction, since W^i is not covering. We have proved that case 2a cannot occur.

We can therefore assume, up to rename the pairs in \mathcal{B}_Π , that there exist meeting components D_1 and D_2 of $\text{Locus}(W^1)$ and $\text{Locus}(W^2)$ such that $D_1 \neq D_2$.

Case 2b $\dim D_1 = \dim D_2 = 4$.

We claim that we cannot have

$$D_1 \cdot V = D_2 \cdot V = 0.$$

In fact, if $D_1 \cdot V = 0$, then for at least a family W^i we have $D_1 \cdot W^i < 0$.

If $i \neq 1, 2$ then $D_1 = \text{Locus}(W^i) = \text{Locus}(W^i)_{\text{Locus}(W_x^1)}$ and $N_1(D_1) = \langle [W^1], [W^i] \rangle$,

hence, by lemma 5.4, $\dim \text{Locus}(W^2)_{D_1} = 5$, a contradiction since W^2 is not covering.

If $D_1 \cdot W^2 < 0$, then $D_2 \subseteq D_1$, against our assumptions, so we have $D_1 \cdot W^1 < 0$ (and, in the same way $D_2 \cdot W^2 < 0$). It follows that $D_1 = \text{Locus}(W^1)$ and $D_2 = \text{Locus}(W^2)$; moreover the locus of every family of a pair belonging to B_Π is contained either in D_1 or in D_2 .

Let $T = D_1 \cup D_2$, let $z \in T$ and let x be a general point of X . Since X is $\text{rc}\mathcal{V}$ connected there exists a chain of cycles in \mathcal{V} connecting x and z ; let Γ be the first irreducible component which meets T .

The curve Γ cannot be numerically proportional to V , since $D_1 \cdot V = D_2 \cdot V = 0$, and its class cannot lie in the plane Π , so Γ belongs to an unsplit family W^i which is independent from W^1 and W^2 ; so either $D_1 \cdot W^i > 0$ or $D_2 \cdot W^i > 0$, which implies that either $D_1 \cdot W^{i+1}$ or $D_2 \cdot W^{i+1}$ is negative, a contradiction which proves the claim.

Therefore we can assume that $D_1 \cdot V > 0$; up to exchange W^3 and W^4 we can also assume that $D_1 \cdot W^3 > 0$.

Let x be a point on a curve in W^4 : then $H = \text{Locus}(W^4, W^3, W^1)_x$ is nonempty and has dimension four by remark 5.5.

If $H \cdot V > 0$ then up to exchange W^5 and W^6 we can assume that $H \cdot W^5 > 0$, and so by lemma 5.4 $\dim \text{Locus}(W^5)_H = 5$, a contradiction.

If $H \cdot V = 0$, for some pair (W^i, W^{i+1}) we have $H \cdot W^i < 0$ and $H \cdot W^{i+1} > 0$.

It cannot be $i = 1$, since in this case $H = \text{Locus}(W^1) = D_1$, but we are assuming that $D_1 \cdot V > 0$; therefore $H \cdot W^i < 0$ for some i such that W^i is independent from W^1 . Let W_H^1 be the closed subfamily of W^1 whose locus is H ; then $H = \text{Locus}(W_H^1)_{\text{Locus}(W_x^i)}$ and so $N_1(H) = \langle [W^1], [W^i] \rangle$.

By construction, $H \cap \text{Locus}(W^3)$ is nonempty, so either $i = 3$, H contains $\text{Locus}(W^3)$ and $\dim \text{Locus}(W^4)_H = 5$, a contradiction, or $i \neq 3$ and $\dim \text{Locus}(W^3)_H = 5$, again a contradiction. So case 2b cannot occur either.

Case 2c $\dim D_1 = 3$.

If D_1 has dimension 3, then D_1 is a component of $\text{Locus}(W_x^1)$ by proposition 2.5; therefore $\dim \text{Locus}(W^2)_{\text{Locus}(W_x^1)} \geq 4$ by lemma 5.4. Let D_2 be a component of $\text{Locus}(W^2)_{\text{Locus}(W_x^1)}$; since W^2 is not covering, D_2 is a divisor in X and, by lemma 4.1, $N_1(D_2) = \langle [W^1], [W^2] \rangle$.

Suppose that $D_2 \cdot V > 0$; then up to exchange W^3 and W^4 we have $D_2 \cdot W^3 > 0$, hence $\text{Locus}(W^3)_{D_2}$ is nonempty and, by lemma 5.4, $\dim \text{Locus}(W^3)_{D_2} = 5$, a contradiction.

So we have $D_2 \cdot V = 0$; in this case D_2 must be negative on one of the W^i , but, since $N_1(D_2) = \langle [W^1], [W^2] \rangle$, $[W^i]$ must belong to Π .

In particular for every pair $(W^i, W^{i+1}) \in \mathcal{B}_\Pi$ we have that $(D_2 \cdot W^i)(D_2 \cdot W^{i+1}) < 0$. Moreover, if $D_2 \cdot W^i < 0$, then $D_2 = \text{Locus}(W^i)$; in fact, if $\dim \text{Locus}(W^i) = 3$ then we can apply lemma 5.4 (a) and get $\dim \text{Locus}(W^2)_{\text{Locus}(W_x^i)} = 5$, a contradiction.

Let T be the union of $\text{Locus}(W^i, W^{i+1})$ for $(W^i, W^{i+1}) \in \mathcal{B}_\Pi$, let $z \in T$ and let x be a point outside T ; since X is $\text{rc}\mathcal{V}$ connected we can join x to z with a chain of cycles in \mathcal{V} ; let Γ be the first irreducible curve in the chain which meets T .

First of all we note that Γ cannot meet D_2 ; in fact, since $D_2 \cdot V = 0$, Γ would be a curve in a family W^i whose class does not lie in the plane Π , so that, by lemma 5.4 $\dim \text{Locus}(W^i)_{D_2} = 5$, a contradiction.

Therefore Γ meets a component $D_i \neq D_2$ of the locus of a family W^i of a pair in \mathcal{B}_Π such that $\text{Locus}(W^{i+1}) = D_2$ and such that $D_2 \cap D_i \neq \emptyset$.

If D_i has dimension four, then we go back to case 2b, so we can assume that $\dim D_i = 3$, i.e. without loss of generality that $D_i = D_1$.

By construction, Γ cannot belong to a family W^i whose class is contained in the plane Π and is not proportional to V ; on the other hand, if Γ belongs to a family W^i whose class is not contained in Π , then, by lemma 5.4 $\dim \text{Locus}(W^i, W^{i+1})_{D_1} = 5$, a contradiction.

It follows that either Γ belongs to an unsplit family αV whose numerical class is proportional to V or Γ belongs to V .

In the first case $\text{Locus}(\alpha V)_{D_1}$ is a divisor D' such that $N_1(D') = \langle [W^1], [\alpha V] \rangle$; if $D' \cdot V > 0$, then we can assume that $\text{Locus}(W^3)_{D'}$ is nonempty and so $\dim \text{Locus}(W^3)_{D'} = 5$, a contradiction.

Therefore $D' \cdot V = 0$, but, since D' meets D_1 and $D' \not\supset D_1$ then $D' \cdot W^1 > 0$ and $D' \cdot W^2 < 0$, so $D' = D_2$ and the curve is contained in T , a contradiction.

Finally, if Γ belongs to V , we use the following

Lemma 9.1. *Let C be an irreducible curve in V . Then either $C \subset \text{Locus}(V_x)$ for some x such that V_x is unsplit or $C \subset \text{Locus}(W^i)$ for some unsplit family W^i such that $[V] = [W^i] + [W^{i+1}]$.*

Proof. If there exists a point $x \in C$ such that V_x is unsplit, then we are in the first case. Otherwise, for every $x \in C$ there passes a reducible cycle $C_x^i + C_x^j \in \mathcal{V}$. Since the families such that $[W^i] + [W^{i+1}] = [V]$ are only a finite number, it follows that $C \subset \text{Locus}(W^i)$ for some i . \square

We thus have two possibilities for Γ : either $\Gamma \subset \text{Locus}(V_x)$, with V_x unsplit, so $\text{Locus}(V_x) \cap D_1 \neq \emptyset$ and therefore $\dim \text{Locus}(V_x) \cap D_1 \geq 1$, a contradiction because $N_1(\text{Locus}(V_x)) = \langle [V] \rangle$ and $N_1(D_1) = \langle [W^1] \rangle$, or $\Gamma \subset \text{Locus}(W^i)$ with $[W^i] \notin \Pi$; in this case $\text{Locus}(W^i, W^{i+1})_{D_1}$ is nonempty and by lemma 5.4 $\dim \text{Locus}(W^i, W^{i+1})_{D_1} = 5$, a contradiction.

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